How sixth grade students explain connections between common and decimal fractions

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Sixth grade students' responses to a task which involved matching common fractions and decimal fractions were examined. Of interest was their understanding of relationships between these symbolic forms, evidenced in explanations offered in an interview setting, and the role of the follow-up question in helping them express their thinking. A most favored explanation involved relating decimal and common fractions to a unit of 100.

Translating between common fractions and decimal fractions has long been accepted as a difficult task for students. According to National Assessment data (Carpenter et al., 1981), 13-year-olds have only about a 50 per cent success rate in converting between even the simplest decimals and fractions—those in which the decimals are powers of ten. Hiebert and Wearne (1986) obtained similar results when they studied approximately 700 fourth- to ninth-graders. When they asked students to write decimals to match pictorial representations of 3/10, 4/100 and 1/5, less than half of even the ninth-grade students studied responded correctly to any of these items. Hiebert and Wearne (1986) attributed these findings to an absence of connections between conceptual and procedural knowledge. They noted:

> In general, students appear to work at a syntactic level that remains isolated from semantics. The two systems of common and decimal fractions are related through manipulation of the written symbols directly rather than through the intermediacy of conceptual reference (p. 209).

Mathematics education for many children is an experience which works to suppress their native and intuitive mathematical concepts. At the age of five or six children commence school with lots of informal and intuitive mathematical concepts and skills. But the schooling process places great value on *the* single answer for evaluation. Much mathematics instruction is authority-based with little space for children to develop intellectual autonomy. Authority-based instruction emphasises the memorisation of rules and procedures at the expense of sharing explanations and meanings. By the time students reach the upper elementary and junior secondary grades, the emphasis from the primary grades on conceptual understanding has been set aside and students become lost in a maze of rules and symbols. Teachers are often appalled when students derive answers that don't make sense either to themselves or their teachers, or are inconsistent with conceptual knowledge previously demonstrated. Teachers assume that since a conceptual base was introduced at some point, students will reference that knowledge as they work through problems. But this doesn't seem to happen.

As students progress through school, symbols used to represent mathematical ideas begin to take on lives of their own, often disconnected to any meaning. Many students no longer expect mathematics to make sense. This is especially evident in the case of fractions and decimals. No one would argue that procedural knowledge needs to be supported by a strong conceptual base. It is just as important though, to have students reference that conceptual base as they work. This leads us to question both the quality of the connections they are making as well as conditions that would enable them to draw upon those connections. In this study we were interested in better understanding: (1) how students understand the connections between common and decimal fraction notational forms, and (2) effectiveness of the follow-up question in an individual interview for enabling students to make such connections.

Method

Forty-nine sixth grade students from four metropolitan Melbourne schools were individually interviewed in November and December of 1995. Twenty three students attended two different private schools, eleven from one school; 12 from the other. Twenty six students attended two government funded schools, fifteen from one and eleven from the other. Classroom teachers were asked to select children whom they considered to be of average or above average ability in mathematics. As such, the sample is representative only of children judged to be successful in primary school mathematics.

These children had almost completed seven years of elementary mathematics education. As an indication of possible curriculum taught, the Victorian Government Department of School Education recommends that children at Level 4 (corresponding approximately to Grades 5 and 6) should be able to

- compare and order decimal fractionds with unequal numbers of places (for example, 3.05, 3.001, 3.4, 3.12);
- count in decimal fractional amounts;
- rename fractions in different forms, including percentages (for example, 250% is the same as 2 1/2); and
- use understood written methods to add and subtract decimal fractions with equal number of places.

Six fractions and decimals tasks were given, as part of a more comprehensive set of 24 tasks representative of the major curriculum topics in mathematics for that age group. The focus of this study is on the sixth task only. A brief description of the six fractions and decimals tasks follows:

Task 1. A flashcard of the fraction 2/3 was presented. The student was asked to draw a diagram to illustrate the fraction.

Task 2. Twelve picture cards were presented to the student. Two-thirds of the cards were to be found.

Task 3. Four flashcards, each showing two fractions, were presented. The student was invited to select pairs whose fractions were equivalent. The pairs were 2/5 and 4/5; 2/5 and 2/8; 4/5 and 8/10; 3/5 and 3/10.

Task 4. A 10x10 grid was presented, together with a flashcard containing the fraction 0.4. The student was asked "If this 10x10 grid was one whole square shade as much of the big square as needed to show this number".

Task 5. Three flashcards each containing the numbers 0.37, 0.123, and 0.62 were presented, and the student asked to identify the largest number.

Task 6. Eight flashcards are presented in random order. The student was asked to match the cards that have the same value. The cards showed these numbers: 1/2, 0.5, 1/4, 0.25, 1/10, 0.1, 2.5, 0.4.

For Task 6 the interviewers were expected to ask follow-up questions in order to better understand the thinking responsible for choices made. The interview protocol contained the general follow-up question "How did you know which cards have the same value?" However the interviewers were free to use their own discretion about what to ask, depending on the students' responses. Often interviewers focused their questions on particular pairs of cards that were chosen by the students.

Results

We will first present results of the kinds of connections students made between the pairs of fractions they selected. Then we will discuss the role of the follow-up question in helping students make connections that might otherwise have remained untapped.

Overall, 34 students (69%) correctly matched all four pairs of cards in their first attempt. Five students whose first response was incorrect changed their choices to correct pairings as a result of further questionning.

The most common way in which these students linked the common and decimal forms of notation was by means of what we will call a *100 scheme*. The 100 scheme can be characterised as a relational device whereby a common fraction or decimal fraction symbol is interpreted in relation to a unit of 100. Thirty two students (65%) were observed to offer justifications that involved reference to 100. Fraction pairs for which the 100 scheme occurred most frequently were 1/4 and 0.25, and 1/2 and 0.5. Justifications for the equivalence of 0.1 and 1/10 that involved references to adding a zero to the one in 0.1, might have either involved a figural correspondence of tens at the symbolic level, or may indeed have been a result of thinking of 0.1 as 10 of 100. We could only make a tentative conjecture based on the quality of explanations for other pairs of choices in this task.

Explanations for matching 0.1 and 1/10. Eight students simply asserted that 0.1 meant one tenth, or tenths, or one tenth of a whole, or the place after the decimal point is tenths. Three students explicitly referred to the digit one being in "the tenths column". As one student explained it: "you have columns, tenths, hundredths, thousandths, and then one (pointing to 0.1) which means one over ten". Two students explained that 0.1 is 10 out of 100. One student offered the argument that 0.1 plus 0.1 added together 10 times would make up a whole. Four explanations were tautological in nature, for example, "that's one tenth and that's one tenth", or, "one tenth is one part of the whole thing, and so is 0.1". Two explanations mentioned adding a zero to the one digit in 0.1: "if you added a zero it would be, like, ten".

Explanations for matching 0.5 and 1/2. The most common explanation involved conceiving 0.5 as half of 100 (10 instances). Typical responses: "one half of 100 is 50 and this equals point five oh", "if you add a zero there's 50 and 50 into 100 is two", and "if you put a zero on the end you get 50, 50 out of 100". There were four explanations in which 0.5 was considered half of 10, as for example, "half is half of anything, point five is half of 10", and "point five, that's out of 10". Five references were made to doubling one of the numbers to make one whole. Examples included "zero point five is five tenths if you doubled it, it would be ten tenths which equals one", "if I added another five that would be one whole", and "point five plus point five equals one". When the interviewer asked one student how he knew 0.5 was one half of a whole number, a geometric explanation was offered: "because it is right in the middle".

Explanations for matching 0.25 and 1/4. Eleven students used arguments that involved 100 and its factors of four and 25. Examples included: "four twenty-fives in one whole is a hundred; 25 is a quarter of a hundred", and "if you times 25 by four it equals a hundred". There were three explanations in which reference to per cent was used, as in "quarter is the same as 25 per cent", and "0.5 is 50 per cent and half of 50 is 25; that's quarter of a hundred". Other explanations also related one fourth as 25 hundredths to one half as 0.5 or 50 hundredths: "half is 50; one quarter is 25", and "0.25 is half of 0.5, and one quarter is half of a half". A similar argument was implicit in the explanation "because 25 plus 25 equals 50, and 50 plus 50 equals 100". Geometric arguments included "if you divide a block of chocolate into quarters color in 25 times 25 times four equals one whole, one hundred". The following arguments seemed to rest on the knowledge that four sub-units comprised the whole: "you've divided the whole by four even if its a decimal", "times it by four you get one", and "both are one quarter, there is four in one whole".

Explanations for matching 0.4 and 2/5. There were three broad categories of explanation used in justifying the matching of 0.4 and 2/5. Explanations based on 100 were observed in seven instances. One example: "two fifths out of 100, 20, there's five twenties, twentieths in 100 or five twos in 10, so um two times that would be four, that's forty". In this example we see the student coordinating unit systems based on 100 and 10.

Other examples: "one fifth of 100 is 20, so two fifths of 100 is 40", and "that (0.4) is 40, and this is 40 as well and this is two fifths and one fifth is 20 into 100, and so two fifths is 40". One explanation was couched in terms of per cent: "one fifth is 20 per cent so times it by two to get two fifths".

The second category of explanation was relationships where one fifth was thought of as two tenths: "two fifths (0.4) 'cause it goes two, four, six, eight, ten", "a fifth as a decimal would be 0.2 but if you times it by two you would have 0.4", and "one fifth would be two, two fifths would be four".

The third category was a more procedural explanation involving the idea that "whatever you do to the top you do to the bottom" or multiplying both numerator and denominator by two to give an equivalent fraction. Eight instances were observed, as for example, "four tenths, bring it down to two fifths", "times that by two halves, if you times it by one it would stay the same", and "times it (2/5) by two halves to give four tenths". One explanation combined both procedural and conceptual elements: "2/5 'cause if you divided 10 into five equal parts, it would be two, and then if you got two fifths, times it by two twos it would be tenths". One explanation involved an overview statement: "they are the same value except they are in different forms".

Follow-up questions. We will examine each of five cases where questions by the interviewer led to modifications to pairings initially selected.

Case 1. This student, Mark, initially made two correct pairs: 0.1 and 1/10, and 0.5 and 1/2, and two incorrect pairs: 2/5 and 0.25, and 0.4 and 1/4. The interviewer began by saying "explain to me why you chose 0.1 and 1/10?"

M: 'Cause that's (tapping the 0.1 card with pen), point one is one tenth. You sort of, you know that...

I: You know that...

M: Yeah.

I: Its just something you know.

M: Yeah.

Her second question: "What about a half and point five?"

M: Well point five is a half of, if there's a one there (pointing to the 0 in the units place) or if you added five onto that it would equal one whole, and half, half and half is one, so...they're the same.

I: Right, okay. Explain to me why you have chosen those two (places 0.4 and 1/4 in front of Mark).

M: They're wrong (reaches for 0.25 card and places it adjacent to 1/4 card).

I: They're wrong. Explain to me why.

M: Out of a hundred, there's, if you divide it equally into four parts, it'd be 25, 25 times four is a hundred, and that's asking for one quarter, one fourth, so and that's 25.

Mark's justifications for his first two choices were adequate though not particularly compelling. Many children found it difficult to express why 1/2 and 0.5 were the same value. Perhaps a reason for this difficulty is because knowledge of these fractions is so basic and fundamental. As ideas become familiar their supporting arguments seem to become less necessary. But clearly the reasons he offered to support his decision to change his third and fourth choices were founded on a solid conceptual basis. The unifying idea was the relationship between 100, 25 and four. The knowledge that he brought forward, which did not feature in the initial selection of pairs, was there nonetheless. This became evident when Mark was asked to explain his reasons for matching up the cards.

Case 2. Jim's initial response was the same as Mark's. He made the following matches to begin with: 0.4 and 1/4, 0.1 and 1/10, 0.25 and 2/5, and 0.5 and 1/2. The interviewer's first question was:

How did you decide that those paired up?

J: 'Cause that's one (pointing to 0.1 card) and that's one tenth (pointing to 1/10 card), and that's four (tapping 0.4 card) 'cause its in tenths, and that's one tenth (pointing to 0.1 card), and that's four tenths (waving pen back and forth between the 0.4 and 1/4 cards), and that's two fifths (pointing to 0.25 card) because its um, 'cause it goes into a hundred, five times (pause), but it goes into it four times, but its, two fifths, and one half (pointing to 0.5 card) is point five, because that's half of 10.
I: Okay so you just remember those do you (pointing to the 0.5 and 1/2 cards)?
J: Yes.
I: So because there's a two and a five in the point two five (pointing to the 0.25 card) that's the same as a two and a five in the fraction (pointing to the 2/5 card)?
J: Yes.
I: And that's the same with the four and one quarter?
J: Actually... (picks up the 2/5 card and pauses for about three seconds)
I: You can change it if you like.
J: (Silence).
I: Tell me what you're thinking?
J: (Picks up the 1/4 card and switches it with the 2/5 card) oh yeah, that would be there, and that would be there, because that's one fourth of a hundred (pointing to the 1/4 card). That's two fifths (pointing to 0.4 card), because it goes two, four, six, eight, ten (extending fingers on left hand in turn as he utters the number sequence). That's two fifths, and that's one fourth (pointing to 1/4 card) because it goes 25, 50, 75, 100.
I: So you are happy with that arrangement now?
J: Yes.

Jim's incorrect choices seemed to be made on the basis of similarity of figural features of the cards. Perhaps emphasis by the interviewer underscoring these surface similarities, in the case of 0.25 and 2/5, provoked a review of the situation. Perhaps the conviction that 0.4 was four tenths did not transfer readily to the fraction 1/4. Perhaps the verbalisation of 0.25 (two fifths) being related to 100 because "it goes into it four times" set up a dissonance that needed to be resolved. Jim, like Mark, had the necessary concepts to support a sound choice. He was able to rectify his mistake by drawing on his own knowledge. He drew on number sequences to do this. He related two fifths and 0.4 to a number sequence corresponding to a count of fifths in a unit of 10. He related 1/4 and 0.25 by verbalising the sequence of counting by 25s to 100. There were four counts.

Case 3. Carla first selected 2/5 and 1/10 as a pair having the same value, but then decided against it. She then selected 0.5 and 1/2, and 0.1 and 1/10, and provided appropriate explanations. She did not match any other cards, so the interviewer asked: What would two fifths be, in tenths?

C: In tenths that would be...
I: How could you write that as tenths?
C: ...One, um, (pause) four zero, nought point four, oh yes (smiles)
I: Got one?
C: (Picks up 2/5 and 0.4 cards and places them aside).
I: Right, so that was four tenths and that was point four (pointing to cards), why are they the same?
C: Because, if I transferred that into a whole (pointing to 2/5), which I would double five to make a whole, and that means I have to double the two as well, to make four.
I: Uh-huh.
C: And so that's marking that, sort of, out of ten. And so that would be four tenths, and that's (pointing to 0.4) done out of ten as well.
I: Oh, okay, so that really is another name for point four is four tenths.
C: Yes.

I: Okay, alright I'll accept that one. Now look at the last one.

C: And I thought there might be a second one, I wasn't sure.

I: Okay, is there any way of working out whether, whether they are the same?

C: Yes, if there is—I'll just put them over here (moves remaining cards in front of her)—there is four 25s in one whole..., in one hundred.

I: Right, So I think you really mean, oh in one hundred, Okay, four 25s in one hundred...

C: Yes. And that's a quarter (pointing to 1/4 card), so that 25 is a quarter of a hundred

I: Right.

C: So that says a quarter, so they both mean the same thing.

Carla seemed to find the interviewer's leading question to be of assistance, since she selected the appropriate card, 0.4, to match 2/5. However her explanation was couched in procedural language akin to the idea of multiplying numerator and denominator by two. What was the whole to which Carla referred? She mentioned the term *whole* twice in this discussion. The other time was when she spoke of four 25s in one whole. In that case she seemed to mean a unit of one hundred. So it is possible that the first time she used this term she was thinking about a unit or whole consisting of 10. Her later reference to "marking that, sort of, out of ten" lends support to the conjecture. Carla was able to link the 1/4 and 0.25 cards by remarking that there were four 25s in 100. Here the 100 scheme was in evidence. Carla seemed to understand the symbol 1/4, and argued that the 0.25 was a quarter of 100, which in turn was a whole unit.

Case 4. Emily gave sound explanations for pairing 0.5 and 1/2, 0.4 and 2/5, and 0.25 and 1/4. She used the 100 scheme for the latter two pairs. For example, 2/5 was 40 out of 100, and 100 was divided by four to give 25, which related to 0.25. Emily did not think 0.1 and 1/10 indicated the same value, to begin with.

I: Now, are they the same?

E: No.

I: Okay, just look at that one for a minute (covers the 1/10 card with her hand). Point one. Can you write point one as a fraction?

E: Yes they are the same actually.

I: Oh, are they?

E: 'Cause you can write point one as a fraction, you write it as one over ten.

The interviewer deliberately focused Emily's attention on the 0.1 card by covering the 1/10 card with her hand. Her prior explanation for 0.5 does not shed much light on her interpretation of 0.1.

I: How did you decide that (0.5) was half of one whole? How do you know that is half of one whole?

E: Because that would usually have a one there (pointing to the zero), but that's point, sort of, minus, no its not minus, its um half of the big number, a whole number

I: So point five is half of a whole, is that what you're telling me?

E: One whole.

In this case it does not seem possible to conclude either way whether Emily did in fact have a meaningful understanding of 0.1, or had interpreted the interviewer's question to indicate that both cards did in fact have the same value.

Case 5. In this case we see the student, Brenda, make adjustments to the initial set of pairs, but there is very little information the interviewer is able to elicit to confirm the revised choices made. However the adjustments were all correct, indicating the possibility that there was relational knowledge which was not forthcoming in the conversation.

Brenda made the following pairs at the beginning: 0.5 and 2/5, 0.1 and 1/10, 0.25 and 1/2, and 0.4 and 1/4. The interviewer asked her why she chose 0.5 and 2/5:

I: Can you remember what you were thinking?
B: Oh, maybe it was (indistinct utterance).
I: Maybe you'd like to change it.
B: I think, ..., I dunno, 'cause that one equals half a whole (pointing to 0.5), and I thought that, that was (pointing to 2/5) one....
I: Okay, is there another card that says half a whole?
B: Oh, here, this one (picks up 1/2 card and places it next to 0.5).
I: Okay, are you happy with that now?
B: Yep.
I: What about this one (pointing to 0.1 and 1/10)? Why did you choose....?
B: Well that's one tenth (pointing to 1/10 card) and there's point one (pointing to 0.1 card).
I: So are they the same?
B: Yep.
I: Okay (points to next pair: 0.25 and 2/5).
B: And, I don't know about this (appears to be looking at 0.4 and 1/4 pair. There is a pause for about three seconds. Brenda picks up the 0.25 card and swaps it with the 0.4 card, so that 0.4 and 2/5 match up, and 0.25 and 1/4 match up). I think that,..., yeah, now they're swapped around.
I: Okay, now why do you think two fifths is the same as point four?
B: Well, 'cause two fifths,... equals, oh point four, I think. 'Cause a quarter equals oh point 25.
I: So you're quite happy with that?
B: Yeah (nodding head).

Discussion

Explanations sixth graders offered for matching up pairs of decimal and common fractions were sought in individual interviews. These explanations revealed a range of different relational connections upon which students' knowledge of fractions and decimals was based. This relational knowledge was closely tied to procedural strategies. A large number of explanations seemed procedural in nature, yet there was evidence of a meaning base present also.

By far the most common mental object used to explain the similarities between common and decimal forms was a 100 scheme. The 100 scheme was observed operating in explanations for all fraction pairs, with the pair 0.1 and 1/10 evoking this explanation the least. A possible explanation for the occurrences of this phenomenon may be found in an earlier task (Task 4), where a 10x10 grid was presented for the student to represent the fraction 0.4. Recent experience of responding to this task may have been remembered. Thus the 100 grid may have served as a context to which both common fraction and decimal fraction symbols were related.

The cases of students who were observed to change their pair choices suggests that opportunities to reflect on a choice, and in addition, to think about how to explain to another individual reasons for such a choice, can have positive outcomes. Students in this study were asked to provide explanations for their choice of pairs as a natural consequence of a clinical interview. Yackel and Cobb (1995) have noted that students who are encouraged to provide explanations to others for their mathematical actions create objects of reflection, both for themselves and for others. In other words they make public their thinking, which can then be tested, accepted, clarified, or queried.

As we have seen, sometimes it can surprise us to learn that a student can make choices based upon superficial task features such as similarities between clusters of digits in flashcard displays, even when they are later found to have quite impressive numerical relations and conceptual schemes for connecting different syntactic systems. However, even when opportunity to review and justify choices is afforded, it is not always possible for students to articulate their reasons for making those choices, as Brenda's case shows. She may well have made lucky guesses on the day, but it is important to consider the possibility that she knew more than she was able to explain. Concluding that a student does not know when in fact he or she does, is a serious problem in assessment and evaluation of student achievement, and indeed in the teaching of mathematics generally. The individual interview is able to reduce this kind of error significantly, but not eliminate it altogether. Opportunities to reflect and justify can sometimes bear little fruit. A number of students in this study who were asked to explain their incorrect choices did not or could not see anything amiss in their responses. It is these students who need further attention. The teacher needs to review with them basic meanings of common fractions and decimal fractions, and interrelationships between them. Physical models such as area grids, and number lines can be useful in this regard. Some experimental work (Case, 1997) that takes children's understanding of per cent as a starting point for instruction in fractions and decimals holds promise.

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